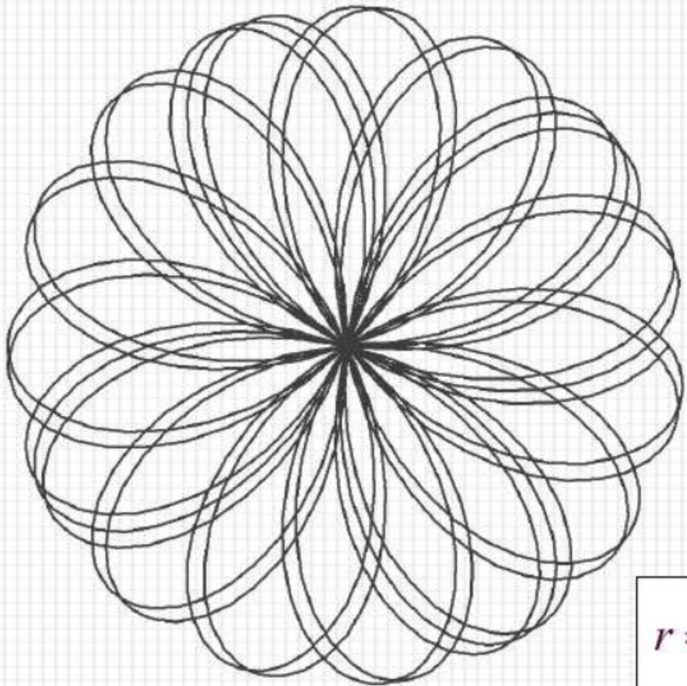
10.6: The Calculus of Polar Curves - Area



$$r = 2\sin(2.15\theta)$$
$$0 \le \theta \le 16\pi$$

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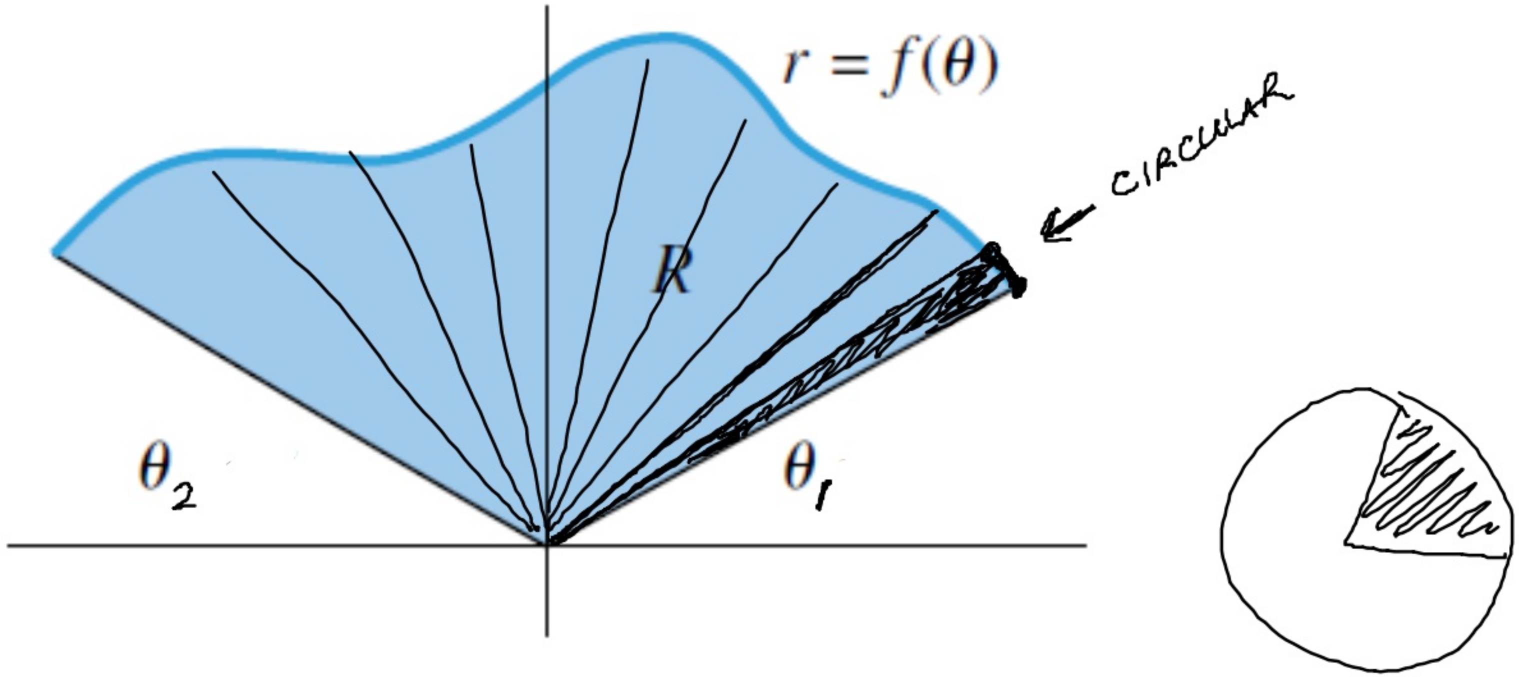
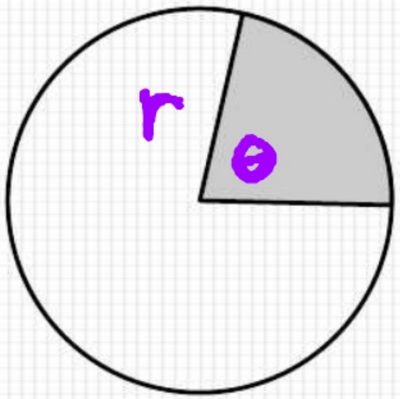


$r = f(\theta)$
CIRCULAR
R
θ_2
θ_1

Area Inside a Polar Graph:

For Cartesian functions, the area under a curve was based on finding the areas of rectangles. Integration added up all the rectangles as they got infinitely thin (Riemann Sum.)

For polar functions, the area within the curve is based on finding the areas of **sectors**:



Area of Sector:

$$A = \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta$$

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Area Inside a Polar Graph:

If we make θ infinitely thin, it becomes dθ: $A = \frac{1}{2} r^2 d\theta$

To find the total area, add up all of the sectors – use integration!

$$\boxed{\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 \, d\theta}$$

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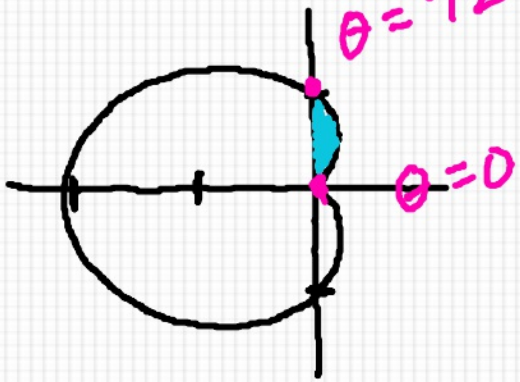
Ex: Find the area enclosed by $r = 1 - \cos\theta$ in quadrant I:



REDUCTION IDENT.

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos\theta)^2\, d\theta$$

$$\frac{1}{2}\int_0^{\pi/2} 1 - 2\cos\theta + \cos^2\theta\, d\theta$$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} 1 - 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\, d\theta$$

$$\frac{1}{2}\left(\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right)\Big]_0^{\pi/2}$$

$$\frac{1}{2}\left(\frac{3}{2}\cdot\frac{\pi}{2} - 2\right) = \boxed{\frac{3\pi}{8} - 1} \approx .178$$

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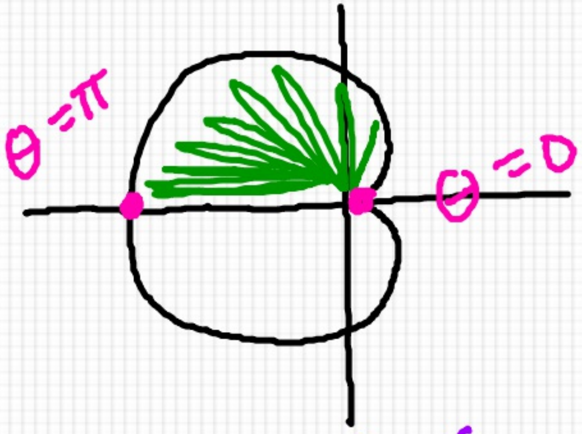
Ex: Find the total area enclosed by $r = 1 - \cos\theta$:

$$\cancel{2}\cdot\cancel{\frac{1}{2}}\int_0^{\pi} (1-\cos\theta)^2\,d\theta$$

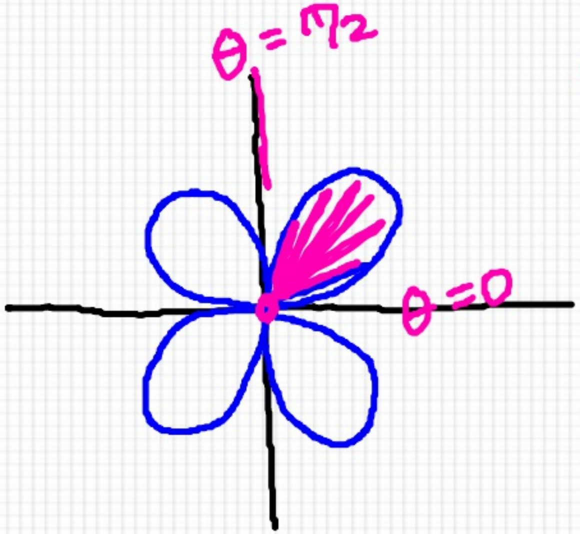


FROM PREVIOUS EXAMPLE

$$\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\Big]_0^{\pi}$$

$$\boxed{\frac{3}{2}\pi}$$

Ex: Find the area enclosed by $r = \sin(2\theta)$:



$$\sin(2\theta) = 0$$
$$(2\theta) = 0, \pi, 2\pi, \dots$$
$$\theta = 0, \pi/2, \dots$$

REDUCTION IDENT.

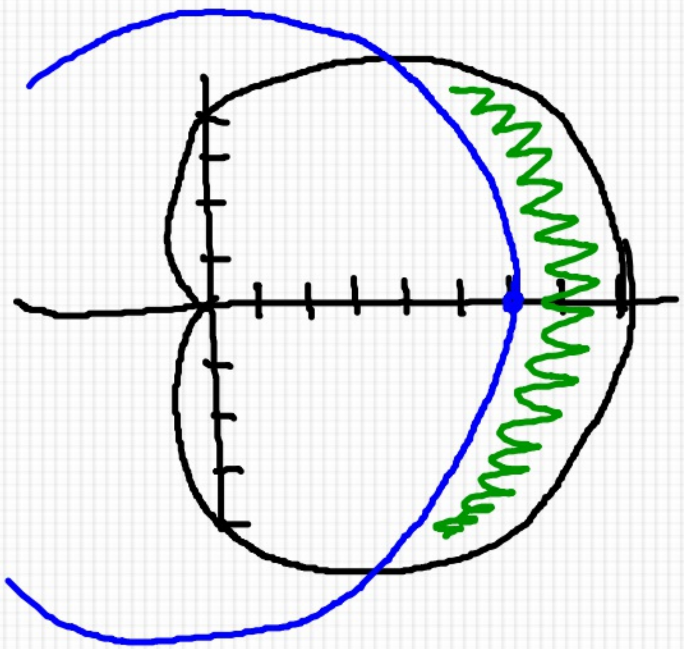
$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$4 \cdot \frac{1}{2} \int_0^{\frac{\pi}{2}} \underbrace{\sin^2(2\theta)} \, d\theta$$

$$\cancel{4} \cdot \cancel{\frac{1}{2}} \int_0^{\frac{\pi}{2}} \cancel{\frac{1}{2}} (1 - \cos(4\theta)) \, d\theta$$

$$\theta - \frac{1}{4}\sin 4\theta \Big]_0^{\pi/2} = \boxed{\frac{\pi}{2}}$$

Ex: Find the area inside of $r = 4 + 4\cos\theta$ and outside of $r = 6$:



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Homework (all noncalculator):

Anton p. 640 #1,2,3,8,9,11,19,20,21,25